Extremely Unfriendly Colourings of ω-regular graphs

Notation

We let $\omega = \{0, 1, 2, \ldots\}$

We write $n \in \omega$ and $n \in \mathbb{N}$ interchangeably

Robin's Question

Let $s_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ switch the n^{th} bit

Do $\exists A, B$ which are Bernoulli$(\frac{1}{2})$-measurable & with $\mu(A) > 0$ s.t. $\forall x \in A$, $\exists \infty$-many $n \in \mathbb{N}$ s.t. exactly one of

$$\{x, s_n(x)\} \in B \,?$$

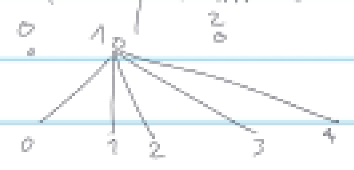
Defn

Let (X, μ) be a std probability space.

Let $G = (X, E)$ a Borel graph,

$E \subseteq X^2$ is Borel. G is not locally finite, but locally countable

Ex



$\mathbb{N}^{\mathbb{N}}$

Ex

H = Hamming graph on $2^{\mathbb{N}}$

$V(H) = 2^{\mathbb{N}}$,

$$x = (a_0, \ldots) \overset{E}{\Longleftrightarrow} y = (b_0, \ldots)$$

if x and y differ by a bit flip

Defn Friendly

We say that a colouring $f: X \to \omega$ is friendly iff $\forall$ a.e. $x \in X$, x has only finitely many neighbours coloured diff from x.

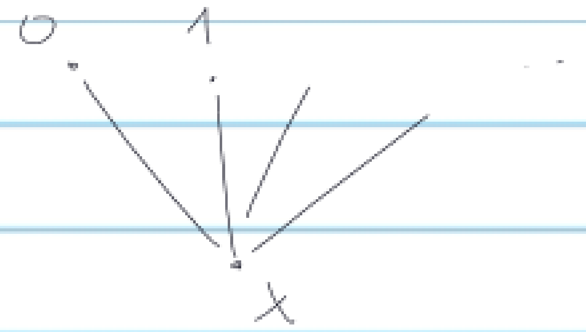
Equivalent Statement:

There exists a μ-measurable 2-colouring of H which is unfriendly.

$\Rightarrow B = \{$blue vertices$\}$

Corollary (Hou, 2022)

There exists a coloring $f: 2^{\mathbb{N}} \to \omega$
s.t. $\forall x \in 2^{\mathbb{N}}$, $f[N_a(x)] = \omega$



$\Rightarrow$ Collapse the colorings into 2 colors.

Defn (K-domatic)

Let $K \le \aleph_0$ a countable cardinal

then a colouring $f: X \to K$ is

K-domatic if $f[N_a(x)] = K \quad \forall x \in X$

Ex

Q_n

$V(Q_n) = \{0,1\}^n$

This has an n-domatic coloring if n is a power of 2.

Q_2  Q_3

Defn

We say $G=(X,E)$ is μ-preserving iff $\exists$ a ctble $\mathcal{A}=\{f_i : X \to X, \text{ Borel, pmp}\}$ s.t. all edges $e \in E(G)$ can be written as

$$(x, f(x)) \text{ for some } f \in \mathcal{A}$$

Ex

$f_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$

Theorem (Hou, 2022)

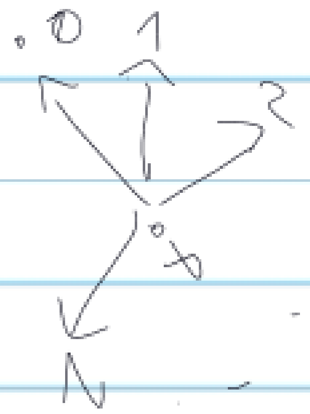
Let G be an ω-regular, μ-preserving Borel graph. Then G admits a μ-meas. ω-domatic coloring.

Prop (Hou, 2022?) 99% lemma

Let G be ω-regular $\forall k \in \mathbb{N}$, there exists a μ-meas. colouring f of G s.t. f is k-dominating on a set of measure $\geq 1-\varepsilon$

Proof sketch

~~Let~~ Use Lusin-Novikov to choose $\forall x \in X$ N neighbours in a Borel way

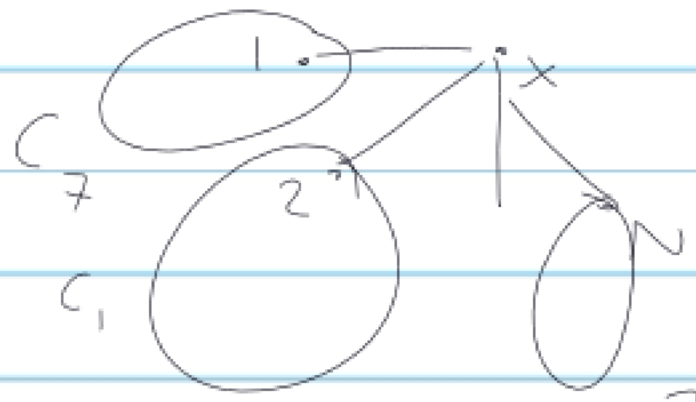


We look at H, the induced subgraph constructed by connecting x to its N neighbours

Then $\exists$ a Borel partition of

$X = \bigsqcup_{n\in\omega} C_n$ and a set A of $\mu(A) > 1-\varepsilon$

s.t. $\forall x \in A$, $N_H(x)$ land different in C_i



look at the space $\{[k]\}^\omega$

$(a_0, a_1, \ldots)$

$C_0 \quad C_1 \quad \ldots$

$\Rightarrow \chi : X \to [k]$

which is (99%) k-chromatic

G admits a μ-measurable ω-chromatic colouring

By the lemma we have $f_n : X \to 2^n$ s.t. f_n is 2^n-chromatic on a set A_n of measure $\geq 1-2^{-n}$

$\{x \in X : \exists^\infty n \; x \in X \setminus A_n\}$ is μ-null by Borel-Cantelli.

$A = \{x \in X : \forall^\infty n \; x \in A_n\}$ $\mu(A) = 1$

for each n take D_n to be the dismal colour (of least measure)
$\mu(D_n) \leq 2^{-n}$
$\{x \in X : \exists^\infty n \; x \in D_n\}$ is μ-null.

Y the set of x lying in only finitely many D_n

for each $x \in A_n$, $\exists y \in N_G(x)$ $y \in D_n$

$g : Y \to [\omega]^{\omega}$

 $\mapsto \{2, 7\}$

Since G is μ-preserving we can take $B \subseteq A \cap Y$
s.t. B is [illegible] and B is E_G-invariant

$g|B$ assigns inf many colors to the nbhd of x

$x \in A$, so there's some n_0 s.t. $\forall n > n_0$ $x \in A_n$ (n_0, ∞)
x has a neighbour in D_n $\forall n > n_0$

$$\bigcup g[N_G(x)] \supseteq (n_0, \infty)$$

$$\bigcup_{y \in N_G(x)} g(y)$$

By E_G invariance $N_G(x) \subseteq B$

each $y \in N_G(x)$ is only contained in finitely many D_n

$\bigcup_{y \in N_G(x)} g(y)$ is infinite and a union of finite sets

So there are inf. many distinct images

It remains to show that $G \upharpoonright B$ admits an ω-chromatic colouring

$r \in \omega^{\omega}$

$rng(x) = \omega$

$\nu_0(\{n\}) = \frac{1}{2^{n+1}}$ $\nu = \prod \nu_0$

$g(x) = A$

for $a \in A$ $r(a) = n$ with probability $\frac{1}{2^{n+1}}$

Let $x \in A$. If x is blue, $x \in B$, $\exists (n_k)_{k \geq 0} \subseteq \mathbb{N}$ s.t.

exactly one $\{x, S_{n_k}(x)\} \in B$

$\Leftarrow$ Let $f : \cancel{X}\, 2^{\mathbb{N}} \longrightarrow \{\text{red}, \text{blue}\}$ an unfriendly colouring. It's unfriendly on A, $\mu(A) > 0$.